Novel solution of Wheeler-DeWitt theory

Łukasz Andrzej Glinka

laglinka@gmail.com

International Institute for Applicable Mathematics & Information Sciences, Hyderabad (India) & Udine (Italy),

B.M. Birla Science Centre, Adarsh Nagar, 500 063 Hyderabad, India

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Abstract

We present a novel solution of the Wheeler–DeWitt equation based on the model resulting due to application of the generalized one-dimensional (1D) conjecture. The conjecture extends the global 1D one on wave functions dependent on both matter fields and a generalized dimension which is a functional of the global one. The residual singularity in the effective potential is eliminating by an appropriate choice of the dimension. Application of the dimensional reduction within the obtained two-component 1D model yields the Dirac equation which is solved in an exact way. By use of the inverted change of variables in this solution we construct a general solution.

Keywords quantum gravity models ; Wheeler–DeWitt equation ; Schrödinger equation ; Dirac equation ; one-dimensionality conjecture

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1 Introduction

The Wheeler–DeWitt theory, well known also as quantum geometrodynamics, is both the historically first and the basic model of quantum gravity considered in modern theoretical physics (See *e.g.* Ref. [1]). Understanding of its physical content, however, is still a great theoretical riddle. Applications to physical phenomena in high energy physics seems to be the mostly interesting. The problem with the model has a mathematical nature, *i.e.* model is given by a functional differential equation with respect to a wave function determined on the Wheeler superspace of on 3-dimensional metrics. Actually, the Wheeler–DeWitt equation was solved for some highly symmetrical classical solutions, and its experimental side is studied [2].

Quantum general relativity arises by employing of the $3+1$ splitting of spacetime metric within the Einstein–Hilbert action supplemented by the York–Gibbons–Hawking boundary term. It leads to the Hamiltonian form of the action, and definition of primary and secondary constraints. One of the secondary constraints, the Hamiltonian contraint, is canonically quantized according to the Dirac–Faddeev method. In result, there is obtained second order functional differential equation on superspace of 3-dimensional embeddings, where the solution is a wave function in general depending on an induced metric and matter fields. The problem, however, is an establishing of any solution of the equation. In spite that there is known a formal path integral solution, the Hartle–Hawking wave function, in general a physical meaning of the solution is not well defined.

This paper reconsiders the Wheeler–DeWitt equation by using of the generalized 1D conjecture, discussed in some aspect in [3], and having sources in generic cosmology [4]. The conjecture is based on reduction of the equation into the Wheeler superspace subset, called DeWitt minisuperspace. The global dimension is an embedding's volume form, and obtained potential is the Wheeler–DeWitt one, with exchange of $\sqrt{h}$ for $2/3h$. The our idea is an application of the change of variables which could regularize the singular character of the potential. The regularization is the generalized dimension being a special functional of the global one. After solution of the received theory, we apply inverted change of variables within the solution, and in result the solution of the Wheeler–DeWitt equation is constructed by a novel method.

Paper is organized as follows. In Section 2 basic established facts are referred. Section 3 presents the conjecture and the change of variables. Dimensional reduction of the model is done in Section 4, and 1D wave function is constructed in Section 5. In Section 6 the general solution is received, and Section 7 briefly discusses all results.

2 Canonical Quantum Gravity

Let us recall the basics of Wheeler–DeWitt theory. General relativity [5], governed by the Einstein field equations (in units $8\pi G/3 = 1$, $c = 1$)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}{}^{(4)}R + \Lambda g_{\mu\nu} = 3T_{\mu\nu}, \tag{1}$$

where Λ is cosmological constant and $T_{\mu\nu}$ is stress-energy tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi[g]}{\delta g^{\mu\nu}} \quad , \quad S_\phi[g] \equiv \int_M d^4x\sqrt{-g}L_\phi, \tag{2}$$

and L_ϕ is Matter fields Lagrangian, models spacetime by a 4-dimensional pseudo–Riemannian manifold (M, g) with a metric $g_{\mu\nu}$, connections $\Gamma^\rho_{\mu\nu}$, curvature tensor $R^\lambda_{\mu\alpha\nu}$, second fundamental form $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$, and scalar curvature ${}^{(4)}R = g^{\kappa\lambda}R_{\kappa\lambda}$. If M is closed and has an induced spacelike boundary $(\partial M, h)$ with a metric h_{ij}, second fundamental form K_{ij}, and an extrinsic curvature $K = h^{ij}K_{ij}$ then (1) arise by variational principle used to the Hilbert action with the York–Gibbons–Hawking term [6]

$$S[g]=\int_M d^4x\sqrt{-g}\left\{-\frac{1}{6}{}^{(4)}R + \frac{\Lambda}{3}\right\} + S_\phi[g] - \frac{1}{3}\int_{\partial M} d^3x\sqrt{h}K \quad . \tag{3}$$

The Nash embedding theorem [7] allows using $3 + 1$ splitting [8]

$$g_{\mu\nu} = \left[\begin{array}{cc} -N^2 + N^iN_i & N_j \\ N_i & h_{ij} \end{array}\right] \quad , \quad h_{ik}h^{kj} = \delta_i^j \quad , \quad N^i = h^{ij}N_j, \tag{4}$$

for which the action (3) takes the canonical form $S[g] = \int dtL$ with

$$L = \int_{\partial M} d^3x\left\{\pi_\phi\dot{\phi} + \pi\dot{N} + \pi^i\dot{N}_i + \pi^{ij}\dot{h}_{ij} - NH - N_iH^i\right\}, \tag{5}$$

where π's are canonical conjugate momenta, and H, H^i are [9]

$$\pi_\phi = \frac{\partial L_\phi}{\partial\dot{\phi}} \quad , \quad \pi = \frac{\partial L}{\partial\dot{N}} \quad , \quad \pi^i = \frac{\partial L}{\partial\dot{N}_i} \quad , \quad \pi^{ij} = \sqrt{h}\left(K^{ij} - Kh^{ij}\right), \tag{6}$$

$$H^i = 2\pi^{ij}{}_{;j} \quad , \quad H = \sqrt{h}\left\{{}^{(3)}R[h] + K^2 - K_{ij}K^{ij} - 2\Lambda - 6\varrho[\phi]\right\}, \tag{7}$$

with ${}^{(3)}R \equiv h^{ij}R_{ij}$, $\varrho[\phi] = n^\mu n^\nu T_{\mu\nu}$, $n^\mu = (1/N)\left[1, -N^i\right]$, and holds

$$\dot{h}_{ij} = 2NK_{ij} + N_{i|j} + N_{j|i}. \tag{8}$$

where $N_{i|j}$ is an intrinsic covariant derivative of N_i. DeWitt [10] showed that H^i are generators of the spatial diffeomorphisms $\widetilde{x}^i = x^i + \xi^i$, *i.e.*

$$i\left[h_{ij}, \int_{\partial M} H_a \xi^a d^3x\right] = -h_{ij,k}\xi^k - h_{kj}\xi^k{}_{,i} - h_{ik}\xi^k{}_{,j} \quad , \tag{9}$$

$$i\left[\pi^{ij}, \int_{\partial M} H_a \xi^a d^3x\right] = -\left(\pi^{ij}\xi^k\right)_{,k} + \pi^{kj}\xi^i{}_{,k} + \pi^{ik}\xi^j{}_{,k} \quad , \tag{10}$$

where $H_i = h_{ij}H^j$, and that the first-class algebra is satisfied

$$i\left[H_i(x), H_j(y)\right] = \int_{\partial M} H_a c^a_{ij} d^3z \quad , \quad i\left[H(x), H_i(y)\right] = H\delta^{(3)}_{,i}(x,y), \tag{11}$$

$$i\left[\int_{\partial M} H\xi_1 d^3x, \int_{\partial M} H\xi_2 d^3x\right] = \int_{\partial M} H^a\left(\xi_{1,a}\xi_2 - \xi_1\xi_{2,a}\right)d^3x. \tag{12}$$

where $c^a_{ij} = \delta^a_i \delta^b_j \delta^{(3)}_{,b}(x,z)\delta^{(3)}(y,z) - (i \leftrightarrow j, x \leftrightarrow y)$ are structure constants of the diffeomorphism group, and all Lie brackets of π's and H's vanish. Time-preservation [11] of the primary constraints, *i.e.* $\pi \approx 0$, $\pi^i \approx 0$, leads to the secondary constraints - scalar (Hamiltonian) and vector respectively

$$H \approx 0 \quad , \quad H^i \approx 0 \quad , \tag{13}$$

Scalar constraint yields dynamics, vector one merely reflects diffeoinvariance. Using the canonical momentum π^{ij} within the scalar constraint yield the Einstein–Hamilton–Jacobi equation (See [12] and some modern studies [13])

$$H = G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{h}\left({}^{(3)}R[h] - 2\Lambda - 6\varrho[\phi]\right) \approx 0 \quad , \tag{14}$$

where $G_{ijkl} \equiv (2\sqrt{h})^{-1}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right)$ is the metric on superspace, a factor space of all C^∞ Riemannian metrics on ∂M, and a group of all C^∞ diffeomorphisms of ∂M that preserve orientation [14]. The Dirac–Faddeev primary canonical quantization method [11, 15]

$$i\left[\pi^{ij}(x), h_{kl}(y)\right] = \frac{1}{2}\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right)\delta^{(3)}(x,y) \quad , \tag{15}$$

$$i\left[\pi^i(x), N_j(y)\right] = \delta^i_j\delta^{(3)}(x,y) \quad , \quad i\left[\pi(x), N(y)\right] = \delta^{(3)}(x,y) \quad , \tag{16}$$

used for the constraint (14) yields the Wheeler–DeWitt equation [12, 10]

$$\left\{G_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} + \sqrt{h}\left({}^{(3)}R[h] - 2\Lambda - 6\varrho[\phi]\right)\right\}\Psi[h_{ij},\phi] = 0 \quad , \tag{17}$$

and other first class constraints merely reflect diffeoinvariance

$$\pi\Psi[h_{ij},\phi] = 0 \quad , \quad \pi^i\Psi[h_{ij},\phi] = 0 \quad , \quad H^i\Psi[h_{ij},\phi] = 0 \quad , \tag{18}$$

and are not important in this model, called quantum geometrodynamics.

3 1D conjecture

3.1 Global dimension

Global one–dimensionality within the quantum General Relativity (17) considered in [3], arises from the change of variables in the Wheeler–DeWitt equation

$$h_{ij} \rightarrow h = \det h_{ij} = \frac{1}{3}\varepsilon^{ijk}\varepsilon^{abc}h_{ia}h_{jb}h_{kc} \quad , \tag{19}$$

where ε^{ijk} is the Levi-Civita density. Using of the Jacobi rule for differentiation of a determinant of a metric $g_{\mu\nu}$ in the 3+1 splitting one obtains

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} \longrightarrow N^2\delta h = N^2 h h^{ij}\delta h_{ij}, \tag{20}$$

and consequently one establishes the differentiation

$$\frac{\delta}{\delta h_{ij}} = h h^{ij}\frac{\delta}{\delta h} \quad . \tag{21}$$

Applying (21) within the quantum geometrodynamics (17) and doing double contraction of the superspace metric with an embedding metric one receives

$$G_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} = -\frac{3}{2}h^{3/2}\frac{\delta^2}{\delta h^2}, \tag{22}$$

so that finally the Wheeler–DeWitt equation (17) can be rewritten as

$$\left(\frac{\delta^2}{\delta h^2} + V_{eff}[h,\phi]\right)\Psi[h,\phi] = 0. \tag{23}$$

Here $V_{eff}[h,\phi]$ is the effective potential

$$V_{eff}[h,\phi] \equiv \frac{2}{3}\frac{{}^{(3)}R}{h} - \frac{4}{3}\frac{\Lambda}{h} - \frac{4}{h}\varrho[\phi]. \tag{24}$$

First term of the potential (24) describes contribution due to an embedding geometry only, the second one is mix of the cosmological constant and an embedding geometry, and the third component is due to Matter fields and an embedding geometry. In result we have to deal with the wave function of a type $\Psi[h,\phi]$, and the basic Wheeler–DeWitt wave function $\Psi[h_{ij},\phi]$ can be reconstructed by inverse change of variables $h \rightarrow h_{ij}$.

3.2 Generalized dimensions

The potential (24) is singular type, which can be eliminated by the general change of variables

$$h \to \xi = \xi[h], \tag{25}$$

$$\delta\xi = \left(\frac{\delta\xi}{\delta h}\right) h h^{ij}\delta h_{ij}, \tag{26}$$

where a generalized dimension $\xi[h]$ is any functional in the global dimension h. With (25) the one-dimensional equation (23) becomes

$$\left\{\left(\frac{\delta\xi}{\delta h}\right)^2 \frac{\delta^2}{\delta\xi^2} + V_{eff}\left[\xi,\phi\right]\right\}\Psi\left[\xi,\phi\right] = 0, \tag{27}$$

so that for all nonsingular cases $\dfrac{\delta\xi}{\delta h} \neq 0$ one writes

$$\left\{\frac{\delta^2}{\delta\xi^2} + V[\xi,\phi]\right\}\Psi\left[\xi,\phi\right] = 0, \tag{28}$$

where $V[\xi,\phi]$ is given by

$$V[\xi,\phi] = \left(\frac{\delta\xi}{\delta h}\right)^{-2} V_{eff}\left[\xi,\phi\right]. \tag{29}$$

In fact the choice of ξ is a kind of the choice of a "gauge", naturally $\xi[h] \equiv h$ is the generic gauge, *i.e.* the case when a generalized dimension becomes the global dimension. Other choices can be generated directly from this case. Note that the following choice

$$\xi = \sqrt{\frac{8}{3}}\sqrt{h}, \tag{30}$$

$$\delta\xi = \sqrt{\frac{2}{3}}\sqrt{h} h^{ij}\delta h_{ij}, \tag{31}$$

cancels the singularity $1/h$ present in the effective potential $V_{eff}\left[h,\phi\right]$ (24), and the equation (28) reads

$$\left\{\frac{\delta^2}{\delta\xi^2} + {}^{(3)}R[\xi] - 2\Lambda - 6\varrho[\phi]\right\}\Psi\left[\xi,\phi\right] = 0, \tag{32}$$

Solving (32) and applying inverse change of variables $\xi \to h_{ij}$ the basic Wheeler–DeWitt wave function $\Psi\left[h_{ij},\phi\right]$ can be reconstructed. The appropriate normalization condition should be chosen as

$$\int \left|\Psi\left[\xi,\phi\right]\right|^2 \delta\mu(\xi,\phi) = 1, \tag{33}$$

where $\mu(\xi,\phi)$ is an invariant measure.

4 Dimensional reduction

Let us chose the product measure $\mu(\xi,\phi)=\delta\xi\delta\phi$. Eq. (28) can be derived as the Euler-Lagrange equation of motion by variational principle $\delta S[\Psi]=0$ applied to the action

$$S[\Psi]=-\frac{1}{2}\int\delta\xi\delta\phi\Psi[\xi,\phi]\left(\frac{\delta^2}{\delta\xi^2}+V[\xi,\phi]\right)\Psi[\xi,\phi]= \tag{34}$$

$$=-\frac{1}{2}\int\delta\phi\Psi[\xi,\phi]\frac{\delta\Psi[\xi,\phi]}{\delta\xi}+\frac{1}{2}\int\delta\xi\delta\phi\left\{\left(\frac{\delta\Psi[\xi,\phi]}{\delta\xi}\right)^2+V[\xi,\phi]\Psi^2[\xi,\phi]\right\} \tag{35}$$

where partial differentiation was used. Choosing the coordinate system so that the boundary term vanishes

$$-\frac{1}{2}\int\delta\phi\Psi[\xi,\phi]\frac{\delta\Psi[\xi,\phi]}{\delta\xi}=0, \tag{36}$$

and using the fact that

$$S[\Psi]\equiv\int\delta\xi\delta\phi L\left[\Psi[\xi,\phi],\delta\Psi[\xi,\phi]/\delta\xi\right], \tag{37}$$

one obtains the Lagrangian characteristic for Euclidean field theory

$$L\left[\Psi[\xi,\phi],\frac{\delta\Psi[\xi,\phi]}{\delta\xi}\right]=\frac{1}{2}\left(\frac{\delta\Psi[\xi,\phi]}{\delta\xi}\right)^2+\frac{V[\xi,\phi]}{2}\Psi^2[\xi,\phi], \tag{38}$$

for which the corresponding canonical conjugate momentum is

$$\Pi_\Psi[\xi,\phi]=\frac{\partial L}{\partial\left(\delta\Psi[\xi,\phi]/\delta\xi\right)}=\frac{\delta\Psi[\xi,\phi]}{\delta\xi}, \tag{39}$$

so that the choice (36) actually means orthogonal coordinates

$$\Psi[\xi,\phi]\Pi_\Psi[\xi,\phi]=0, \tag{40}$$

for any values of ξ and ϕ. With using of the relation (39) the Eq. (28) can be rewritten in the form

$$\frac{\delta\Pi_\Psi[\xi,\phi]}{\delta\xi}+V[\xi,\phi]\Psi[\xi,\phi]=0, \tag{41}$$

and its combination with the Eq. (39) yield the appropriate Dirac equation

$$\left(i\gamma\frac{\delta}{\delta\xi}-M[\xi,\phi]\right)\Phi[\xi,\phi]=0, \tag{42}$$

where we have employed the notation

$$\Phi[\xi,\phi]=\left[\begin{array}{c}\Pi_\Psi[\xi,\phi]\\ \Psi[\xi,\phi]\end{array}\right]\quad,\quad M[\xi,\phi]=\left[\begin{array}{cc}1&0\\0&V[\xi,\phi]\end{array}\right],\tag{43}$$

and the γ-matrices algebra consists only one element - the Pauli matrix σ_y

$$\gamma=\left[\begin{array}{cc}0&-i\\ i&0\end{array}\right]\equiv\sigma_y\quad,\quad\gamma^2=I,\tag{44}$$

where I is the identity matrix, that in itself obey the algebra

$$\{\gamma,\gamma\}=2\delta_E\quad,\quad\delta_E=\left[\begin{array}{cc}1&0\\0&1\end{array}\right].\tag{45}$$

Dimensional reduction of the one component second order theory (28) yields the two component first order one (42) possessing the Clifford algebra of Euclidean type [16] $\mathcal{C}\ell_{1,1}(\mathbb{R})$ that is the matrix algebra possessing a complex 2-dimensional representation. Restricting to $Pin_{1,1}(\mathbb{R})$ yield a 2D spin representations; restricting to $Spin_{1,1}(\mathbb{R})$ splits it onto a sum of two 1D Weyl representations; $\mathcal{C}\ell_{1,1}(\mathbb{R})$ decomposes into a direct sum of two isomorphic central simple algebras or a tensor product

$$\mathcal{C}\ell_{1,1}(\mathbb{R})=\mathcal{C}\ell^+_{1,1}(\mathbb{R})\oplus\mathcal{C}\ell^-_{1,1}(\mathbb{R})=\mathcal{C}\ell_{2,0}(\mathbb{R})\otimes\mathcal{C}\ell_{0,0}(\mathbb{R}),\tag{46}$$

$$\mathcal{C}\ell_{1,1}(\mathbb{R})\cong\mathbb{R}(2)\quad,\quad\mathcal{C}\ell^\pm_{1,1}(\mathbb{R})=\frac{1\pm\gamma}{2}\mathcal{C}\ell_{1,1}(\mathbb{R})\cong\mathbb{R}\quad,\quad\mathcal{C}\ell_{0,0}(\mathbb{R})\cong\mathbb{R}.\tag{47}$$

5 1D wave function

The Dirac equation (42) can be rewritten in the Schrödinger form

$$i\frac{\delta\Phi[\xi,\phi]}{\delta\xi}=H[\xi,\phi]\Phi[\xi,\phi]\quad,\quad H[\xi,\phi]=i\left[\begin{array}{cc}0&-V[\xi,\phi]\\1&0\end{array}\right].\tag{48}$$

Solution of the evolution (48) can be written as

$$\Phi[\xi,\phi]=U[\xi,\phi]\Phi[\xi^I,\phi],\tag{49}$$

where $\Phi[\xi^I,\phi]$ is an initial data vector with respect to ξ only, and $U[\xi,\phi]$ is unitary evolution operator

$$U=\exp\left\{-i\int_{\Sigma(\xi)}\delta\xi' H[\xi',\phi]\right\}=\exp\left\{-i\Omega(\xi,\phi)\langle H\rangle(\xi,\phi)\right\},\tag{50}$$

where $\Sigma(\xi)$ is finite integration area in ξ-space, Ω is the volume of full configuration space, and $\langle H\rangle(\phi)$ is an averaged energy given by the formulas

$$\Omega(\xi,\phi)=\int_{\Sigma(\xi,\phi)}\delta\xi'\delta\phi' \quad , \quad \langle H\rangle(\xi,\phi)=\frac{1}{\Omega(\xi,\phi)}\int_{\Sigma(\xi)}\delta\xi' H[\xi',\phi]. \tag{51}$$

where $\Sigma(\xi,\phi)=\Sigma(\xi)\times\Sigma(\phi)$ is finite integration region of full configurational space. Explicitly one obtains

$$U[\xi,\phi]=\mathbf{1}_2\cosh\left[\Omega(\xi,\phi)\sqrt{\langle V\rangle(\xi,\phi)}\right]+ \tag{52}$$

$$+\left[\begin{array}{cc} 0 & \sqrt{\langle V\rangle(\xi,\phi)} \\ \left(\sqrt{\langle V\rangle(\xi,\phi)}\right)^{-1} & 0 \end{array}\right]\sinh\left[\Omega(\xi,\phi)\sqrt{\langle V\rangle(\xi,\phi)}\right], \tag{53}$$

with

$$\langle V\rangle(\xi,\phi)=\frac{1}{\Omega(\xi,\phi)}\int_{\Sigma(\xi)}\delta\xi' V[\xi',\phi]. \tag{54}$$

Elementary algebraic manipulations yield the generalized one-dimensional wave function as

$$\begin{aligned}\Psi[\xi,\phi] &= \Psi[\xi^I,\phi]\cosh\left[\Omega(\xi,\phi)\sqrt{\langle V\rangle(\xi,\phi)}\right]+ \\ &+ \Pi_\Psi[\xi^I,\phi]\left(\sqrt{\langle V\rangle(\xi,\phi)}\right)^{-1}\sinh\left[\Omega(\xi,\phi)\sqrt{\langle V\rangle(\xi,\phi)}\right], \end{aligned}\tag{55}$$

and the canonical conjugate momentum as the solution is

$$\begin{aligned}\Pi_\Psi[\xi,\phi] &= \Pi_\Psi[\xi^I,\phi]\cosh\left[\Omega(\xi,\phi)\sqrt{\langle V\rangle(\xi,\phi)}\right]+ \\ &+ \Psi[\xi^I,\phi]\sqrt{\langle V\rangle(\xi,\phi)}\sinh\left[\Omega(\xi,\phi)\sqrt{\langle V\rangle(\xi,\phi)}\right], \end{aligned}\tag{56}$$

where $\Psi[\xi^I,\phi]$ and $\Pi_\Psi[\xi^I,\phi]$ are initial data with respect to ξ only. Applying, however, the equation (39) for (56) one obtains the relation

$$\Pi_\Psi[\xi,\phi]=\frac{\Pi_\Psi[\xi^I,\phi]}{\sqrt{\langle V\rangle}}\frac{\delta}{\delta\xi}\left[\Omega\sqrt{\langle V\rangle}\right]\cosh\left[\Omega\sqrt{\langle V\rangle}\right]+$$

$$+\left[\Psi[\xi^I,\phi]\frac{\delta}{\delta\xi}\left[\Omega\sqrt{\langle V\rangle}\right]+\Pi_\Psi[\xi^I,\phi]\frac{\delta}{\delta\xi}\left[\left(\sqrt{\langle V\rangle}\right)^{-1}\right]\right]\sinh\left[\Omega\sqrt{\langle V\rangle}\right], \tag{57}$$

where for shorten notation $\Omega \equiv \Omega(\xi,\phi)$ and $\langle V\rangle \equiv \langle V\rangle(\xi,\phi)$, which after calculation of the functional derivatives

$$\begin{aligned}\frac{\delta}{\delta\xi}\left[\Omega\sqrt{\langle V\rangle}\right] &= \frac{1}{2}\sqrt{\langle V\rangle}\left(\frac{\delta\Omega}{\delta\xi}+1\right), &(58)\\ \frac{\delta}{\delta\xi}\left[\left(\sqrt{\langle V\rangle}\right)^{-1}\right] &= \frac{1}{2}\left[\Omega\sqrt{\langle V\rangle}\right]^{-1}\left(\frac{\delta\Omega}{\delta\xi}-1\right), &(59)\end{aligned}$$

and using it within the formula (57) yields

$$\Pi_\Psi[\xi,\phi] = \Pi_\Psi[\xi^I,\phi]\frac{1}{2}\left(\frac{\delta\Omega}{\delta\xi}+1\right)\cosh\left[\Omega\sqrt{\langle V\rangle}\right] + \\ +\left[\Psi[\xi^I,\phi]\frac{\sqrt{\langle V\rangle}}{2}\left(\frac{\delta\Omega}{\delta\xi}+1\right)+\frac{\Pi_\Psi[\xi^I,\phi]}{2\Omega\sqrt{\langle V\rangle}}\left(\frac{\delta\Omega}{\delta\xi}-1\right)\right]\sinh\left[\Omega\sqrt{\langle V\rangle}\right]. \quad (60)$$

After comparison with (56) one obtains the system of equations

$$\left\{\begin{array}{c}\frac{1}{2}\left(\frac{\delta\Omega}{\delta\xi}+1\right)=1,\\ \\ \Psi[\xi^I,\phi]\frac{1}{2}\left(\frac{\delta\Omega}{\delta\xi}+1\right)+\frac{\Pi_\Psi[\xi^I,\phi]}{\Omega\langle V\rangle}\frac{1}{2}\left(\frac{\delta\Omega}{\delta\xi}-1\right)=\Psi[\xi^I,\phi]\end{array}\right. . \quad (61)$$

The first equation of the system (61) yields the relation

$$\frac{\delta\Omega}{\delta\xi} = 1 = \int_{\Sigma(\phi)}\delta\phi', \quad (62)$$

where the last integral arises by the first formula in (51), which after application to the second equation gives the self-consistent identity $\Psi[\xi^I,\phi] = \Psi[\xi^I,\phi]$. It means also that the volume $\Omega(\xi,\phi)$ is ϕ-invariant, *i.e.*

$$\Omega(\xi,\phi) = \int_{\Sigma(\xi)}\delta\xi' = \Omega(\xi). \quad (63)$$

Directly from (55) the probability density can be deduced easily as

$$\begin{aligned}|\Psi[\xi,\phi]|^2 &= (\Psi[\xi^I,\phi])^2\cosh^2\left[\Omega\sqrt{\langle V\rangle}\right] + \\ &+ (\Pi_\Psi[\xi^I,\phi])^2(\langle V\rangle)^{-1}\sinh^2\left[\Omega\sqrt{\langle V\rangle}\right] + \\ &+ \Psi[\xi^I,\phi]\Pi_\Psi[\xi^I,\phi]\left(\sqrt{\langle V\rangle}\right)^{-1}\sinh\left[2\Omega\sqrt{\langle V\rangle}\right], \quad (64)\end{aligned}$$

and in the light of the relation (40) it simplifies to

$$|\Psi[\xi,\phi]|^2 = (\Psi[\xi^I,\phi])^2\cosh^2\left[\Omega\sqrt{\langle V\rangle}\right] + (\Pi_\Psi[\xi^I,\phi])^2\left(\langle V\rangle\right)^{-1}\sinh^2\left[\Omega\sqrt{\langle V\rangle}\right]. \tag{65}$$

Putting by hands the following separation conditions

$$\Psi[\xi^I,\phi] = \Psi[\xi^I]\Gamma_\Psi[\phi] \quad , \quad \Pi_\Psi[\xi^I,\phi] = \Pi_\Psi[\xi^I]\Gamma_\Pi[\phi], \tag{66}$$

where Γ_Ψ and Γ_Π are functionals of ϕ only and $\Psi[\xi^I]$, and $\Pi_\Psi[\xi^I]$ are constant functionals, and applying the usual normalization one obtains the simple constraint

$$\int_{\Sigma(\xi,\phi)} |\Psi[\xi',\phi']|^2\delta\xi'\delta\phi' = 1 \longrightarrow A(\Pi_\Psi[\xi^I])^2 + B(\Psi[\xi^I])^2 - 1 = 0, \tag{67}$$

where the constants A and B are given by the integrals

$$A = \int_{\Sigma(\xi,\phi)} \Gamma_\Pi[\phi']\left(\langle V'\rangle\right)^{-1}\sinh^2\left[\Omega'\sqrt{\langle V'\rangle}\right]\delta\xi'\delta\phi', \tag{68}$$

$$B = \int_{\Sigma(\xi,\phi)} \Gamma_\Psi[\phi']\cosh^2\left[\Omega'\sqrt{\langle V'\rangle}\right]\delta\xi'\delta\phi', \tag{69}$$

which in our assumption are convergent and finite. The equation (67), however, can be solved straightforwardly. In result one obtains the relation

$$\Pi_\Psi[\xi^I] = \pm\sqrt{\frac{1}{A} - \frac{B}{A}(\Psi[\xi^I])^2}, \tag{70}$$

which together with $\Pi_\Psi[\xi^I,\phi] = \dfrac{\delta\Psi[\xi^I,\phi]}{\delta\xi^I}$ and (66 yields differential equation

$$\frac{1}{\Gamma[\phi]}\frac{\delta\Psi[\xi^I]}{\delta\xi^I} = \pm\sqrt{\frac{1}{A} - \frac{B}{A}(\Psi[\xi^I])^2}, \tag{71}$$

where $\Gamma[\phi] \equiv \dfrac{\Gamma_\Pi[\phi]}{\Gamma_\Psi[\phi]}$, which can be integrated

$$\sqrt{A}\int\frac{\delta\Psi[\xi^I]}{\sqrt{1 - B(\Psi[\xi^I])^2}} = \pm\Gamma[\phi]\xi^I + C, \tag{72}$$

where C is a constant of integration, with the result

$$\sqrt{A/B}\arcsin\left\{\sqrt{B/A}\Psi[\xi^I]\right\} = \pm\Gamma[\phi]\xi^I + C, \tag{73}$$

so that after elementary algebraic manipulations one obtains

$$\Psi[\xi^I] = \sqrt{A/B}\sin\theta(\xi^I,\phi), \tag{74}$$

where

$$\theta(\xi^I,\phi) = \sqrt{B/A}\left(\pm\Gamma[\phi]\xi^I + C\right), \tag{75}$$

However, because $\Psi[\xi^I]$ must be a functional of ξ^I only, must holds $\Gamma[\phi] = \Gamma_0$, where Γ_0 is a constant, for which $\theta(\xi^I,\phi) = \theta(\xi^I)$. Taking into account the relation (70) one obtains finally

$$\Psi[\xi^I] = \sqrt{A/B}\sin\theta(\xi^I) \quad , \quad \Pi_\Psi[\xi^I] = \pm\sqrt{\frac{1}{A} - \sin^2\theta(\xi^I)}. \tag{76}$$

In the light of the equation (40), however, must holds one of the relations

$$\sin\theta(\xi^I) \equiv 0 \quad , \quad \sin\theta(\xi^I) = \pm\sqrt{1/A}. \tag{77}$$

The first relation in (77) means that

$$\sqrt{B/A}\left(\pm\Gamma_0\xi^I + C\right) = k\pi \longrightarrow \xi^I = \pm\frac{1}{\Gamma_0}\left(\sqrt{A/B}k\pi - C\right), \tag{78}$$

where $k \in \mathbb{Z}$ is an integer. Similarly the second relation in (77) gives

$$\xi^I = \pm\frac{1}{\Gamma_0}\left(\pm\sqrt{A/B}\arcsin\sqrt{1/A} - C\right). \tag{79}$$

For the first case one has

$$\Psi[\xi^I] = 0 \quad , \quad \Pi_\Psi[\xi^I] = \pm\sqrt{1/A}, \tag{80}$$

and for the second one hold

$$\Psi[\xi^I] = \pm\sqrt{1/B} \quad , \quad \Pi_\Psi[\xi^I] = 0. \tag{81}$$

Finally we see that the generalized one-dimensional wave function (55) is

$$\Psi[\xi,\phi] = \pm\Gamma_\Psi[\phi]\Gamma_0\sqrt{\frac{1}{A}}\left(\sqrt{\langle V\rangle(\xi,\phi)}\right)^{-1}\sinh\left[\Omega(\xi)\sqrt{\langle V\rangle(\xi,\phi)}\right], \tag{82}$$

in the first case of (77), and for the second one

$$\Psi[\xi,\phi] = \pm\Gamma_\Psi[\phi]\sqrt{\frac{1}{B}}\cosh\left[\Omega(\xi)\sqrt{\langle V\rangle(\xi,\phi)}\right]. \tag{83}$$

6 General solution

The general solutions of the Wheeler–DeWitt equation (17) can be now constructed immediately from the generalized one-dimensional solutions (82) and (83) by putting in the integrals

$$\Omega(\xi)=\int_{\Sigma(\xi,\phi)}\delta\xi' \quad , \quad \langle V\rangle(\xi,\phi)=\frac{1}{\Omega(\xi)}\int_{\Sigma(\xi)}\delta\xi' V[\xi',\phi], \tag{84}$$

the ξ-measure following form combination of the relations (30) and (31)

$$\delta\xi=\sqrt{\frac{2}{3}}\sqrt{h}h^{ij}\delta h_{ij}. \tag{85}$$

Because, however, the potential $V[\xi,\phi]$ has a form

$$V[\xi,\phi]={}^{(3)}R[\xi]-2\Lambda-6\varrho[\phi], \tag{86}$$

one has nice separability

$$\langle V\rangle(\xi,\phi)=\frac{1}{\Omega(\xi)}\int_{\Sigma(\xi)}\delta\xi'\ {}^{(3)}R[\xi']-2\Lambda-6\rho[\phi], \tag{87}$$

so that in fact for a concrete 3-dimensional embedding we should estimate the functional average of the 3-dimensional Ricci scalar

$$\langle{}^{(3)}R[h]\rangle=\frac{1}{\Omega(h_{ij})}\int_{\Sigma(h_{ij})}\delta h'_{ij}\sqrt{\frac{2}{3}}\sqrt{h'}h^{ij\prime}\ {}^{(3)}R[h'], \tag{88}$$

where

$$\Omega(h_{ij})=\int_{\Sigma(h_{ij})}\delta h'_{ij}\sqrt{\frac{2}{3}}\sqrt{h'}h^{ij\prime}, \tag{89}$$

which yields the functional average of the potential

$$\langle V\rangle(h_{ij},\phi)=\langle{}^{(3)}R[h]\rangle-2\Lambda-6\rho[\phi]. \tag{90}$$

Using the formula (90) within the solutions (82) and (83) one obtains the general solutions of the Wheeler–DeWitt equation due to the 1D conjecture

$$\Psi[h_{ij},\phi]=\pm\Gamma_{\Psi}[\phi]\Gamma_0\sqrt{\frac{1}{A}}\left(\sqrt{\langle V\rangle(h_{ij},\phi)}\right)^{-1}\sinh\left[\Omega(h_{ij})\sqrt{\langle V\rangle(h_{ij},\phi)}\right], \tag{91}$$

$$\Psi[h_{ij},\phi]=\pm\Gamma_{\Psi}[\phi]\sqrt{\frac{1}{B}}\cosh\left[\Omega(h_{ij})\sqrt{\langle V\rangle(h_{ij},\phi)}\right]. \tag{92}$$

Here the constants A and B are defined as the integrals

$$A = \sqrt{\frac{2}{3}}\Gamma_0 \int_{\Sigma(h_{ij},\phi)} \Gamma_\Psi[\phi'] \frac{\sinh^2\left[\Omega(h'_{ij})\sqrt{\langle V\rangle(h'_{ij},\phi')}\right]}{\langle V\rangle(h'_{ij},\phi')} \sqrt{h'} h^{ij\prime} \delta h'_{ij} \delta\phi', \tag{93}$$

$$B = \sqrt{\frac{2}{3}} \int_{\Sigma(h_{ij},\phi)} \Gamma_\Psi[\phi'] \cosh^2\left[\Omega(h'_{ij})\sqrt{\langle V\rangle(h'_{ij},\phi')}\right] \sqrt{h'} h^{ij\prime} \delta h'_{ij} \delta\phi', \tag{94}$$

and assumed to be convergent and finite. Using for the solutions (91) and (92) the usual normalization condition

$$\int_{\Sigma(h_{ij},\phi)} |\Psi[h_{ij},\phi]|^2 \sqrt{\frac{2}{3}} \sqrt{h'} h^{ij\prime} \delta h'_{ij} \delta\phi = 1, \tag{95}$$

leads to the relations

$$|\Gamma_\Psi[\phi]\Gamma_0|^2 = 1 \quad , \quad \Gamma_\Psi[\phi]\Gamma_0 = 1, \tag{96}$$

which yield $\Gamma_\Psi[\phi] = 1/\Gamma_0$, $\Gamma_0 = 1$, so that finally one obtains

$$\Psi_1[h_{ij},\phi] = \pm\sqrt{\frac{1}{A}} \left(\sqrt{\langle V\rangle(h_{ij},\phi)}\right)^{-1} \sinh\left[\Omega(h_{ij})\sqrt{\langle V\rangle(h_{ij},\phi)}\right], \tag{97}$$

$$\Psi_2[h_{ij},\phi] = \pm\sqrt{\frac{1}{B}} \cosh\left[\Omega(h_{ij})\sqrt{\langle V\rangle(h_{ij},\phi)}\right], \tag{98}$$

where now

$$A = \sqrt{\frac{2}{3}} \int_{\Sigma(h_{ij},\phi)} \frac{\sinh^2\left[\Omega(h'_{ij})\sqrt{\langle V\rangle(h'_{ij},\phi')}\right]}{\langle V\rangle(h'_{ij},\phi')} \sqrt{h'} h^{ij\prime} \delta h'_{ij} \delta\phi', \tag{99}$$

$$B = \sqrt{\frac{2}{3}} \int_{\Sigma(h_{ij},\phi)} \cosh^2\left[\Omega(h'_{ij})\sqrt{\langle V\rangle(h'_{ij},\phi')}\right] \sqrt{h'} h^{ij\prime} \delta h'_{ij} \delta\phi'. \tag{100}$$

The solutions (97) and (101) describe two independent states in the quantum gravity model given by the Wheeler–DeWitt equation (17).

Because, however, the equation (17) is linear, in general the superposition

$$\Psi[h_{ij},\phi] = \sum_{i=1,2} \alpha_i \Psi_i[h_{ij},\phi] \tag{101}$$

where α_i are arbitrary constants, is its a general solution. In the light of the normalization condition (95), it means that the constraint holds

$$|\alpha_1|^2 + |\alpha_2|^2 + (\alpha_1^\star\alpha_2 + \alpha_1\alpha_2^\star) I = 1, \tag{102}$$

where

$$I=\sqrt{\frac{1}{AB}}\int_{\Sigma(h_{ij},\phi)}\frac{\sinh\left[2\Omega(h'_{ij})\sqrt{\langle V\rangle(h'_{ij},\phi')}\right]}{2\sqrt{\langle V\rangle(h'_{ij},\phi')}}\sqrt{\frac{2}{3}}\sqrt{h'}h^{ij\prime}\delta h'_{ij}\delta\phi'. \quad (103)$$

For vanishing $I=0$ one obtains form (102) simply

$$|\alpha_2|=\sqrt{1-|\alpha_1|^2} \quad , \quad |\alpha_1|\geqslant 1. \quad (104)$$

The case of $I\neq 0$ is more complicated. Note that (102) can be rewritten in form

$$(\alpha_1+\alpha_2 I)\alpha_1^\star+(\alpha_2+\alpha_1 I)\alpha_2^\star=0\longrightarrow\frac{\alpha_1^\star}{\alpha_2^\star}=\frac{-\alpha_1 I+\alpha_2}{\alpha_1+\alpha_2 I}, \quad (105)$$

or in the equivalent form

$$C\alpha_1^\star=-\alpha_1 I+\alpha_2 \quad , \quad C\alpha_2^\star=\alpha_1+\alpha_2 I, \quad (106)$$

where $0\neq C\in\mathbb{R}$. The relations (106) establish the absolute values on

$$C|\alpha_1|^2=-\alpha_1^2 I+\alpha_2\alpha_1 \quad , \quad C|\alpha_2|^2=\alpha_1\alpha_2+\alpha_2^2 I, \quad (107)$$

which after mutual adding and using of (102) yields the equation

$$CI[(\alpha_1^\star-\alpha_2)\alpha_2+(\alpha_2^\star+\alpha_1)\alpha_1]=\alpha_1\alpha_2+\alpha_2\alpha_1, \quad (108)$$

which gives the relations

$$CI(\alpha_1^\star-\alpha_2)=\alpha_1 \quad , \quad CI(\alpha_2^\star+\alpha_1)=\alpha_2. \quad (109)$$

Using of the complex decomposition for α and α_2 within (109) leads to

$$\Re\alpha_2=(CI-1)\Re\alpha_1 \quad , \quad \Im\alpha_2=(CI-1)\Im\alpha_1, \quad (110)$$

or equivalently

$$\alpha_2=(CI-1)\alpha_1 \quad , \quad |\alpha_2|^2=(CI-1)^2|\alpha_1|^2. \quad (111)$$

Employing (111) within the constraint (102) yields to

$$|\alpha_1|^{-2}=IC^2+(I^2-2I)C-I+2. \quad (112)$$

Because, however, both $|\alpha_i|^2\in\mathbb{R}$ as squares of absolute values, one obtains the region of values of the constant C in dependence on the integral I

$$C\in[-\infty,C_-]\cup[C_+,\infty] \quad , \quad C_\pm=\frac{I-2}{2}\left[1\pm\sqrt{1-\frac{4}{I(I-2)}}\right], \quad (113)$$

where for $C_\pm\in\mathbb{R}$ the condition $I\in[-\infty,1-\sqrt{5}]\cup[1+\sqrt{5},\infty]$ holds.

7 Outlook

This paper has discussed the selected consequence arising due to application of the generalized one-dimensional conjecture within the Wheeler–DeWitt quantum geometrodynamics. We have shown that employing the conjecture immediately yield construction of a general solution. The obtained formulation in general uses the Lebesgue–Stieltjes 1D integrals. There are open problems related to the novel wave functions. Especially, black holes exploration by the presented method seems to be intriguing. Similarly discussion of non vanishing cosmological constant, and conformal flat classical solutions are interesting. The other problem is generalization of the results for the case of D-branes.

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